10.15 Figure 1 displays the binomial tree that describes the value of the put options as a function of the stock price. At each node, the upper number refers to the stock price, whereas figures within parentheses and brackets correspond to the prices of the European and American put options, respectively. The risk-neutral probability p of an upward movement is

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{\frac{1}{4}0.12} - 0.9}{1.1 - 0.9} = 0.6523$$

given that the risk free rate of return is 12%. As the value of the European option at node A coincides with the present value of the expected payoff, it follows that

$$\begin{aligned} f &= \left[f_{uu}\, p^2 + 2\, f_{ud}\, p\,(1-p) + f_{dd}\,(1-p)^2\right]\, e^{-\frac{1}{2}r} \\ &= \left[2\,(42 - 39.6) \times 0.6523 \times 0.3477 + (42 - 32.4) \times 0.3477^2\right]\, e^{-\frac{1}{2}0.12} = 2.118 \end{aligned}$$

Needless to say, the same result holds if one works back the value of the option at node A through the tree. As for the American put option, it has a greater value than the European option because it is optimal to exercise early at node C. To appreciate that, it suffices to observe that the value of the European put option at node C is

$$\begin{aligned} f_d &= \Big[f_{ud}\, p + f_{dd}\,(1-p)\Big]\, e^{-\frac{1}{4}r} \\ &= \Big(2.4 \times 0.6523 + 9.6 \times 0.3477\Big)\, e^{-\frac{1}{4}0.12} = 4.759, \end{aligned}$$

whereas the payoff from early exercise is $K - S = 42 - 36 = 6$.

10.18 The risk-neutral probability of a upward movement is

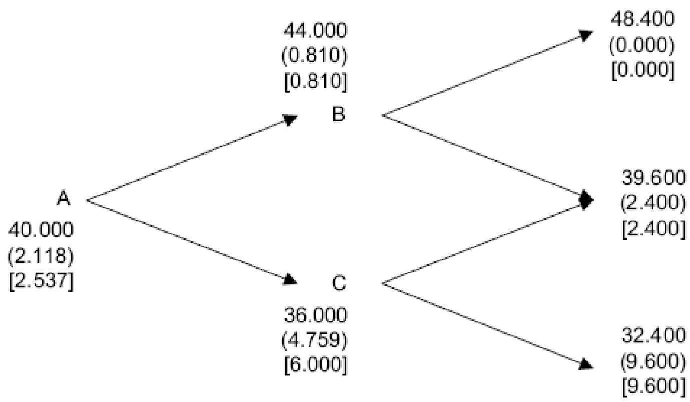
$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{\frac{1}{12}0.05} - 0.9}{1.08 - 0.9} = 0.5788,$$

hence it follows from the present value of the expected payoff at node A in Figure 2 that the value of the power option is

$$\begin{aligned} f &= \left[f_{uu}\, p^2 + 2\, f_{ud}\, p\,(1-p) + f_{dd}\,(1-p)^2\right]\, e^{-\frac{1}{6}r} \\ &= \left[2\,(30 - 29.16)^2 \times 0.5788 \times 0.4212 + (30 - 24.3)^2 \times 0.4212^2\right]\, e^{-\frac{1}{6}0.05} = 6.0587 \end{aligned}$$

As before, the same result holds if one backs out the value of the power option by working back through the binomial tree below. Finally, the result also holds even for the American options in view that, at node C, the payoff from early exercise is $(K - S)^2 = (30 - 27)^2 = 9$, which is less

Figure 1



than the value of the analogous European power option, namely

$$\begin{aligned} f_d &= \left[f_{ud}\, p + f_{dd}\,(1-p)\right] e^{-\frac{1}{12} r} \\ &= \left[(30-29.16)^2 \times 0.5788 + (30-24.3)^2 \times 0.4212\right] e^{-\frac{1}{12} 0.05} = 14.0346 \end{aligned}$$

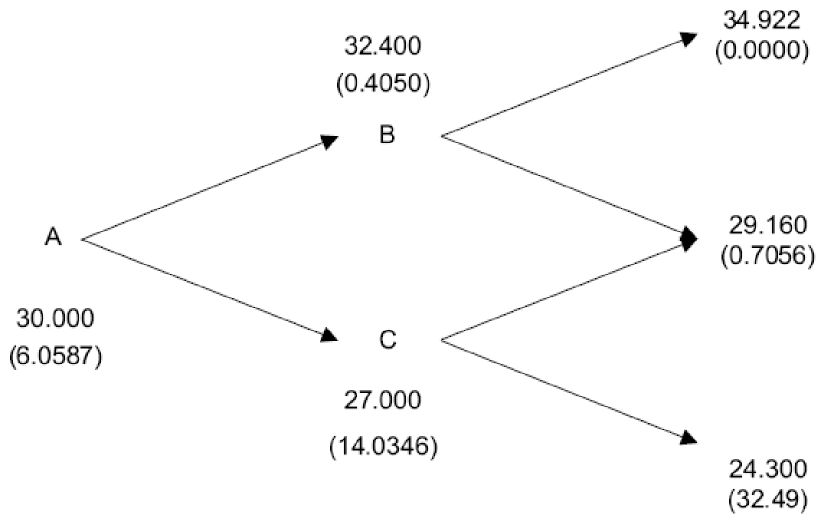
10.19 The binomial tree has two steps and time to maturity of the option is 6 months, hence each step will last for three months, i.e., $\delta t = 1/4$. As the volatility is 30% per annum, it follows that $u = e^{\sigma \sqrt{\delta t}} = e^{\frac{1}{2} 0.3} = 1.1618$ and $d = 1/u = 0.8607$, resulting in a risk-neutral probability of a upward movement of

$$p = \frac{e^{r\,T} - d}{u-d} = \frac{e^{\frac{1}{3} 0.04} - 0.8607}{1.1618 - 0.8607} = 0.4959$$

The value of the option then is 3.3739 as calculated by DerivaGem (select Equity as the underlying type and Binomial European as the option type). If one uses 5, 50, 100 and 500 time steps, DerivaGem will compute the value of the option as 3.9229, 3.7394, 3.7478, and 3.7545, respectively.

11.13 If the cash position follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month with an initial position of 2, then the probability distribution of the cash position after k months is normal with mean $2{+}0.1\,k$ and variance $0.16\,k$. The probability of a random sample from a normal distribution with mean 2.6 and variance 0.96 being negative is $\Phi\left(-\frac{2+6\times 0.1}{\sqrt{6\times 0.16}}\right) = 0.0040$, where Φ is the cumulative distribution function of the standard normal. As for the one-year horizon, the probability of negative cash position is $\Phi\left(-\frac{2+12\times 0.1}{\sqrt{12\times 0.16}}\right) = 0.0107$. Finally, to find at what time the probability of a negative cash flow is greatest, it suffices to minimize

Figure 2



$\frac{2+0.1\,k}{\sqrt{0.16\,k}}$ with respect to k. The first derivative is zero when $k = 20$ months (with positive second derivative).

11.14 By Ito's lemma, the bond price will follow

$$\mathrm{d}B = \left[\frac{\partial B}{\partial x}\, a\,(x_0 - x) + \frac{\partial B}{\partial t} + \frac{\partial^2 B}{\partial x^2}\, s^2\, x^2\right] \mathrm{d}t + \frac{\partial B}{\partial x}\, s\, x\, \mathrm{d}z.$$

The price of a perpetual bond that continuously pays interest at the rate of \$1 per annum is $B = \int_0^\infty e^{-x\,t}\,\mathrm{d}t = 1/x$. It then follows that $\frac{\partial B}{\partial x} = -x^{-2}$, $\frac{\partial B}{\partial t} = 0$, and $\frac{\partial^2 B}{\partial x^2} = 2\,x^{-3}$. The stochastic process of the bond price thus is

$$\mathrm{d}B = \left[-a\,\frac{x_0 - x}{x^2} + \frac{s^2}{x}\right] \mathrm{d}t - \frac{s}{x}\,\mathrm{d}z,$$

which means that the overall expected instantaneous return (i.e., interest plus the expected capital gains) is $1 - a\,\frac{x_0 - x}{x^2} + \frac{s^2}{x}$.